RECOGNISING ACHIEVEMENT

ADVANCED GCE

4764/01

MATHEMATICS (MEI)

Mechanics 4

WEDNESDAY 18 JUNE 2008

Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (24 marks)

1 A rocket in deep space starts from rest and moves in a straight line. The initial mass of the rocket is m_0 and the propulsion system ejects matter at a constant mass rate k with constant speed u relative to the rocket. At time t the speed of the rocket is v.

(i) Show that while mass is being ejected from the rocket, $(m_0 - kt)\dfrac{dv}{dt} = uk$. [5]

(ii) Hence find an expression for v in terms of t. [4]

(iii) Find the speed of the rocket when its mass is $\frac{1}{3}m_0$. [3]

2 A car of mass m kg starts from rest at a point O and moves along a straight horizontal road. The resultant force in the direction of motion has power P watts, given by $P = m(k^2 - v^2)$, where $v\,\mathrm{m\,s^{-1}}$ is the velocity of the car and k is a positive constant. The displacement from O in the direction of motion is x m.

(i) Show that $\left(\dfrac{k^2}{k^2 - v^2} - 1\right)\dfrac{dv}{dx} = 1$, and hence find x in terms of v and k. [9]

(ii) How far does the car travel before reaching 90% of its terminal velocity? [3]

Section B (48 marks)

3 A circular disc of radius a m has mass per unit area $\rho\,\mathrm{kg\,m^{-2}}$ given by $\rho = k(a + r)$, where r m is the distance from the centre and k is a positive constant. The disc can rotate freely about an axis perpendicular to it and through its centre.

(i) Show that the mass, M kg, of the disc is given by $M = \frac{5}{3}k\pi a^3$, and show that the moment of inertia, $I\,\mathrm{kg\,m^2}$, about this axis is given by $I = \frac{27}{50}Ma^2$. [9]

For the rest of this question, take $M = 64$ and $a = 0.625$.

The disc is at rest when it is given a tangential impulsive blow of 50 N s at a point on its circumference.

(ii) Find the angular speed of the disc. [4]

The disc is then accelerated by a constant couple reaching an angular speed of $30\,\mathrm{rad\,s^{-1}}$ in 20 seconds.

(iii) Calculate the magnitude of this couple. [3]

When the angular speed is $30\,\mathrm{rad\,s^{-1}}$, the couple is removed and brakes are applied to bring the disc to rest. The effect of the brakes is modelled by a resistive couple of $3\dot{\theta}$ N m, where $\dot{\theta}$ is the angular speed of the disc in $\mathrm{rad\,s^{-1}}$.

(iv) Formulate a differential equation for $\dot{\theta}$ and hence find $\dot{\theta}$ in terms of t, the time in seconds from when the brakes are first applied. [7]

(v) By reference to your expression for $\dot{\theta}$, give a brief criticism of this model for the effect of the brakes. [1]

4 A uniform smooth pulley can rotate freely about its axis, which is fixed and horizontal. A light elastic string AB is attached to the pulley at the end B. The end A is attached to a fixed point such that the string is vertical and is initially at its natural length with B at the same horizontal level as the axis. In this position a particle P is attached to the highest point of the pulley. This initial position is shown in Fig. 4.1.

The radius of the pulley is a, the mass of P is m and the stiffness of the string AB is $\frac{mg}{10a}$.

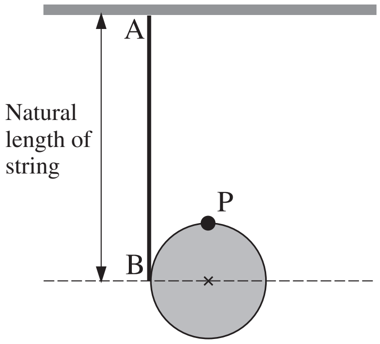

Fig. 4.1

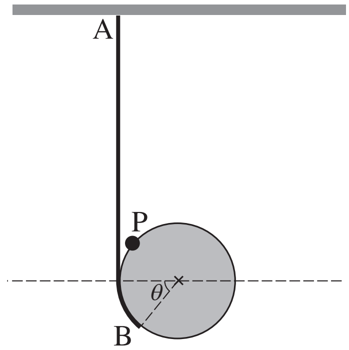

Fig. 4.2

(i) Fig. 4.2 shows the system with the pulley rotated through an angle θ and the string stretched. Write down the extension of the string and hence find the potential energy, V, of the system in this position. Show that $\frac{dV}{d\theta} = mga\left(\frac{1}{10}\theta - \sin\theta\right)$. [6]

(ii) Hence deduce that the system has a position of unstable equilibrium at $\theta = 0$. [6]

(iii) Explain how your expression for V relies on smooth contact between the string and the pulley. [2]

Fig. 4.3 shows the graph of the function $f(\theta) = \frac{1}{10}\theta - \sin\theta$.

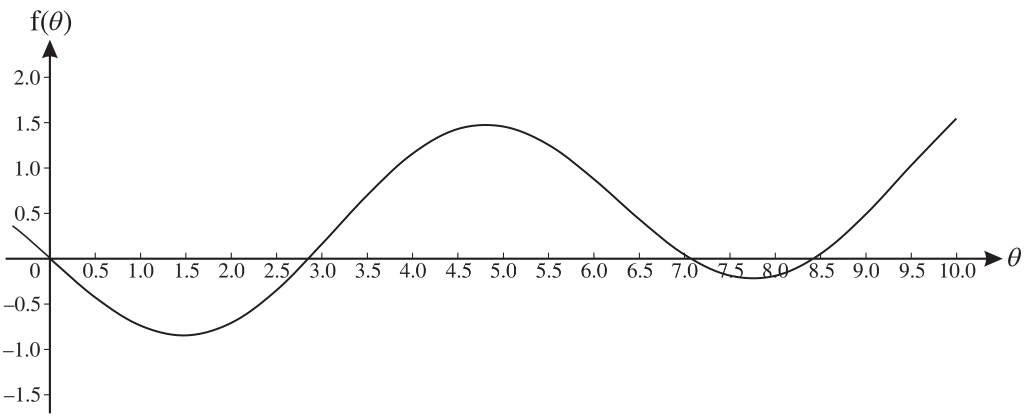

Fig. 4.3

(iv) Use the graph to give rough estimates of three further values of θ (other than $\theta = 0$) which give positions of equilibrium. In each case, state with reasons whether the equilibrium is stable or unstable. [6]

(v) Show on a sketch the physical situation corresponding to the least value of θ you identified in part **(iv)**. On your sketch, mark clearly the positions of P and B. [2]

(vi) The equation $f(\theta) = 0$ has another root at $\theta \approx -2.9$. Explain, with justification, whether this necessarily gives a position of equilibrium. [2]

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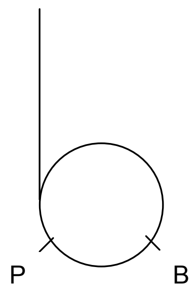
1(i)	If δm is change in mass over time δt PCLM $mv = (m+\delta m)(v+\delta v) + \lvert\delta m\rvert(v-u)$ [N.B. $\delta m < 0$]	M1	Change in momentum over time δt	
	$(m+\delta m)\frac{\delta v}{\delta t} + u\frac{\delta m}{\delta t} = 0 \Rightarrow m\frac{dv}{dt} = -u\frac{dm}{dt}$	M1 A1	Rearrange to produce DE Accept sign error	
	$\frac{dm}{dt} = -k \Rightarrow m = m_0 - kt$	M1	Find m in terms of t	
	$\Rightarrow (m_0 - kt)\frac{dv}{dt} = uk$	E1	Convincingly shown	5
(ii)	$v = \int \frac{uk}{m_0 - kt}dt$	M1	Separate and integrate	
	$= -u\ln(m_0 - kt) + c$	A1	cao (allow no constant)	
	$t = 0, v = 0 \Rightarrow c = u\ln m_0$	M1	Use initial condition	
	$v = u\ln\left(\frac{m_0}{m_0 - kt}\right)$	A1	All correct	4
(iii)	$m = \frac{1}{3}m_0 \Rightarrow m_0 - kt = \frac{1}{3}m_0$	M1 A1	Find expression for mass or time Or $t = 2m_0/3k$	
	$\Rightarrow v = u\ln 3$	A1		3

2(i)	$P = Fv$	M1	Used, not just quoted	
	$= mv\frac{dv}{dx}v$	M1	Use N2L and expression for acceleration	
	$\Rightarrow mv^2\frac{dv}{dx} = m\left(k^2 - v^2\right)$	A1	Correct DE	
	$\Rightarrow \frac{v^2}{k^2 - v^2}\frac{dv}{dx} = 1$	M1	Rearrange	
	$\Rightarrow \left(\frac{k^2}{k^2 - v^2} - 1\right)\frac{dv}{dx} = 1$	E1	Convincingly shown	
	$\int\left(\frac{k^2}{k^2 - v^2} - 1\right)dv = \int dx$	M1	Separate and integrate	
	$\frac{1}{2}k\ln\left(\frac{k+v}{k-v}\right) - v = x + c$	A1	LHS	
	$x = 0, v = 0 \Rightarrow c = 0$	M1	Use condition	
	$x = \frac{1}{2}k\ln\left(\frac{k+v}{k-v}\right) - v$	A1	cao	9
(ii)	Terminal velocity when acceleration zero $\Rightarrow v = k$	M1 A1		
	$v = 0.9k \Rightarrow x = \frac{1}{2}k\ln\left(\frac{1.9}{0.1}\right) - 0.9k = \left(\frac{1}{2}\ln 19 - 0.9\right)k \approx$ 0.572k	F1	Follow their solution to (i)	3

3(i)	$M=\int_0^a k(a+r)2\pi r\,\mathrm{d}r$	M1 M1	Use circular elements (for *M* or *I*) Integral for mass	
	$=2k\pi\left[\frac{1}{2}ar^2+\frac{1}{3}r^3\right]_0^a$	M1 A1	Integrate (for *M* or *I*) For […]	
	$=\frac{5}{3}k\pi a^3$	E1		
	$I=\int_0^a k(a+r)2\pi r\cdot r^2\,\mathrm{d}r$	M1	Integral for *I*	
	$=2k\pi\left[\frac{1}{4}ar^4+\frac{1}{5}r^5\right]_0^a$	A1	For […]	
	$=\frac{9}{10}k\pi a^5$	A1	cao	
	$=\frac{27}{50}Ma^2$	E1	Complete argument (including mass)	9
(ii)	$I=13.5$ $0.625\times 50=I\omega$ $\Rightarrow \omega\approx 2.31$	B1 M1 M1 A1	Seen or used (here or later) Use angular momentum Use moment of impulse cao	4
(iii)	$\ddot{\theta}=\dfrac{30-2.31}{20}\approx 1.38$	M1	Find angular acceleration	
	Couple $=I\ddot{\theta}$ ≈ 18.7	M1 F1	Use equation of motion Follow their ω and *I*	3
(iv)	$I\ddot{\theta}=-3\dot{\theta}$	B1	Allow sign error and follow their *I* (but not *M*)	
	$I\dfrac{\mathrm{d}\dot{\theta}}{\mathrm{d}t}=-3\dot{\theta}$	M1	Set up DE for $\dot{\theta}$ (first order)	
	$\int\dfrac{\mathrm{d}\dot{\theta}}{\dot{\theta}}=\int -\dfrac{3}{I}\mathrm{d}t$	M1	Separate and integrate	
	$\ln\lvert\dot{\theta}\rvert=-\dfrac{t}{4.5}+c$	B1	$\ln(\text{multiple of }\dot{\theta})$ seen	
	$\dot{\theta}=A\mathrm{e}^{-t/4.5}$	M1	Rearrange, dealing properly with constant	
	$t=0,\dot{\theta}=30\Rightarrow A=30$	M1	Use condition on $\dot{\theta}$	
	$\dot{\theta}=30\mathrm{e}^{-t/4.5}$	A1		7
(v)	Model predicts $\dot{\theta}$ never zero in finite time.	B1		1

4(i)	$V=\frac{1}{2}\left(\frac{mg}{10a}\right)(a\theta)^2+mga\cos\theta$ (relative to centre of pulley)	M1	EPE term	
		B1 M1 A1	Extension $=a\theta$ GPE relative to any zero level (± constant)	
	$\frac{\mathrm{d}V}{\mathrm{d}\theta}=\frac{1}{2}\left(\frac{mg}{10a}\right)\cdot 2a^2\theta-mga\sin\theta$	M1	Differentiate	
	$\frac{\mathrm{d}V}{\mathrm{d}\theta}=mga\left(\frac{1}{10}\theta-\sin\theta\right)$	E1		6
(ii)	$\theta=0\Rightarrow\frac{\mathrm{d}V}{\mathrm{d}\theta}=mga\left(\frac{1}{10}(0)-\sin 0\right)=0$	M1	Consider value of $\frac{\mathrm{d}V}{\mathrm{d}\theta}$	
	hence equilibrium	E1		
	$\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}=mga\left(\frac{1}{10}-\cos\theta\right)$	M1 A1	Differentiate again	
	$V''(0)=-0.9mga<0$	M1	Consider sign of V''	
	hence unstable	E1	V'' must be correct	6
(iii)	If the pulley is smooth, then the tension in the string is constant.	B1		
	Hence the EPE term is valid.	B1		2
(iv)	Equilibrium positions at $\theta=2.8$, $\theta=7.1$ and $\theta=8.4$	B1 B1	One correct All three correct, no extras Accept answers in [2.7,3.0), [7,7.2], [8.3,8.5]	
	From graph, $V''(2.8)=mga\mathrm{f}'(2.8)>0$	M1	Consider sign of V'' or f'	
	hence stable at $\theta=2.8$ $V''(7.1)=mga\mathrm{f}'(7.1)<0\Rightarrow$ unstable at $\theta=7.1$ $V''(8.4)=mga\mathrm{f}'(8.4)>0\Rightarrow$ stable at $\theta=8.4$	A1 A1 A1	Accept no reference to V'' for one conclusion but other two must relate to sign of V'', not just f'.	6
(v)		B1 B1	P in approximately correct place B in approximately correct place	2
(vi)	If $\theta<0$ then expression for EPE not valid hence not necessarily an equilibrium position.	M1 A1		2

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General Comments

Many candidates demonstrated a good understanding of Mechanics and high levels of algebraic competency. Approximately half of the candidates found the work on rotation difficult, and for these candidates question 3 was clearly their weakest question.

Comments on Individual Questions

1) (i) Most candidates realised that they had to consider momentum over a small increment in time (rather than simply write down Newton's second law). However, many had sign inconsistencies between the small change in mass and the rate of change of mass.

(ii) This was often done very well, although some candidates made errors in their integration. Some made errors when calculating the arbitrary constant and a few even omitted the constant.

(iii) There were many correct solutions.

2) (i) Most candidates were able to set up a correct differential equation, although some jumped to the printed version with insufficient working. Solving the differential equation presented some problems, but many candidates correctly used the standard result as given in the formula book.

(ii) Most candidates realised that at the terminal velocity the acceleration is zero and correctly substituted the value found into the expression for x.

3) (i) Finding the expression for the mass was usually done well, although a few candidates did not know how to set up the appropriate integral. Most candidates who found the mass went on to find the moment of inertia correctly.

(ii) There were many candidates who did not realise that they needed to use angular momentum. Of those who did, most equated the change in angular momentum to impulse, rather than moment of impulse.

(iii) Some candidates attempted to use energy, without success. Many did use the equation of motion, but some made errors when calculating the acceleration.

(iv) Some candidates produced good solutions for this part. Some were unable to solve their differential equation. Some attempted to set up and solve a second order equation for θ (which is beyond the requirements of the specification). A few could not even write down an appropriate equation of motion. Some unsuccessfully attempted to use energy.

(v) Of those who made some progress with the previous part, many realised that the model predicted that the angular velocity never reaches zero.

4) (i) Most candidates completed this correctly, but some seemed to be led by the printed answer and others mistakenly thought the given expression was the potential energy.

(ii) This was often correct, but sometimes lacked clarity, in particular not giving clear conclusions based on the relevant working.

(iii) Very few candidates appreciated what this part was asking. Most argued about the effect of friction on the whole system. The question specifically asked about the expression for V, and very few realised that the tension in the string would not be constant, hence invalidating the EPE term.

(iv) Many correct solutions were seen.

(v) The sketch was often done well.

(vi) There were many good answers seen, but many wrongly assumed that a negative θ meant that the string was slack.